CS 260: Foundations of Data Science

Prof. Thao Nguyen

Fall 2025

HAVERFORD COLLEGE

Materials by Sara Mathieson

Admin

- **Sit somewhere new**

- **Lab 1** grades & feedback posted on Moodle

Cost Function (mini-quiz)

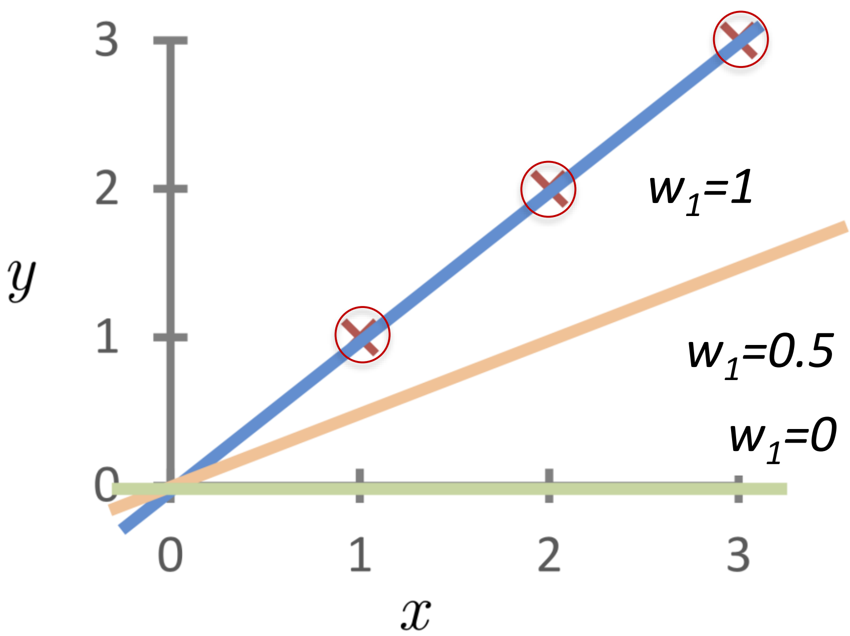
$$h_w(x) = w_1 x$$

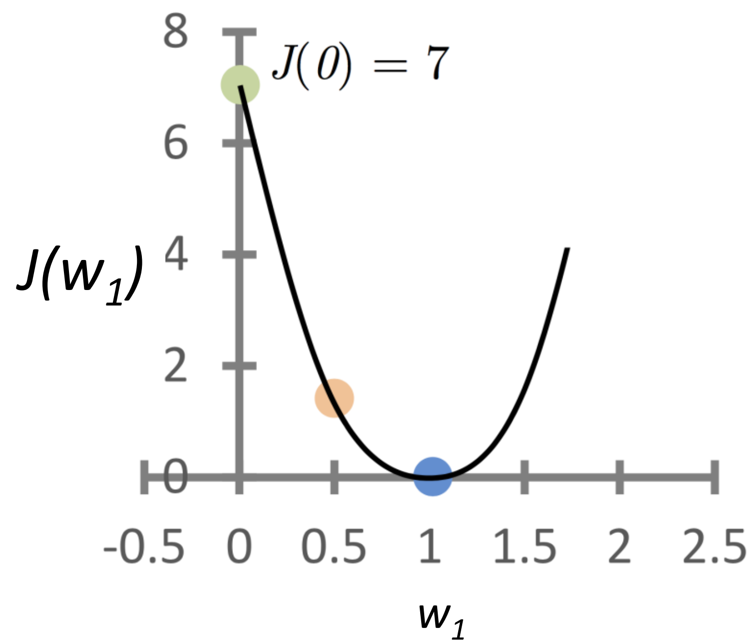
(assume $w_0 = 0$ for this example)

1. What is the cost function for this model?

2. Given the datapoints $(x_1, y_1) = (1,1)$, $(x_2, y_2) = (2,2)$, $(x_3, y_3) = (3,3)$, compute J(0), J(0.5), and J(1)

Cost Function (mini-quiz)





$$J(0.5) = \frac{1}{2}\left[(0.5-1)^2 + (1-2)^2 + (1.5-3)^2\right] = 1.75$$

Slide: modified from Jessica Wu
Based on slide by Eric Eaton
[Example by Andrew Ng]

Outline for today

- SGD (Stochastic Gradient Descent)
- Handout 6 (SGD solution example)
- Analytic vs. SGD (pros and cons)
- (if time) Polynomial regression

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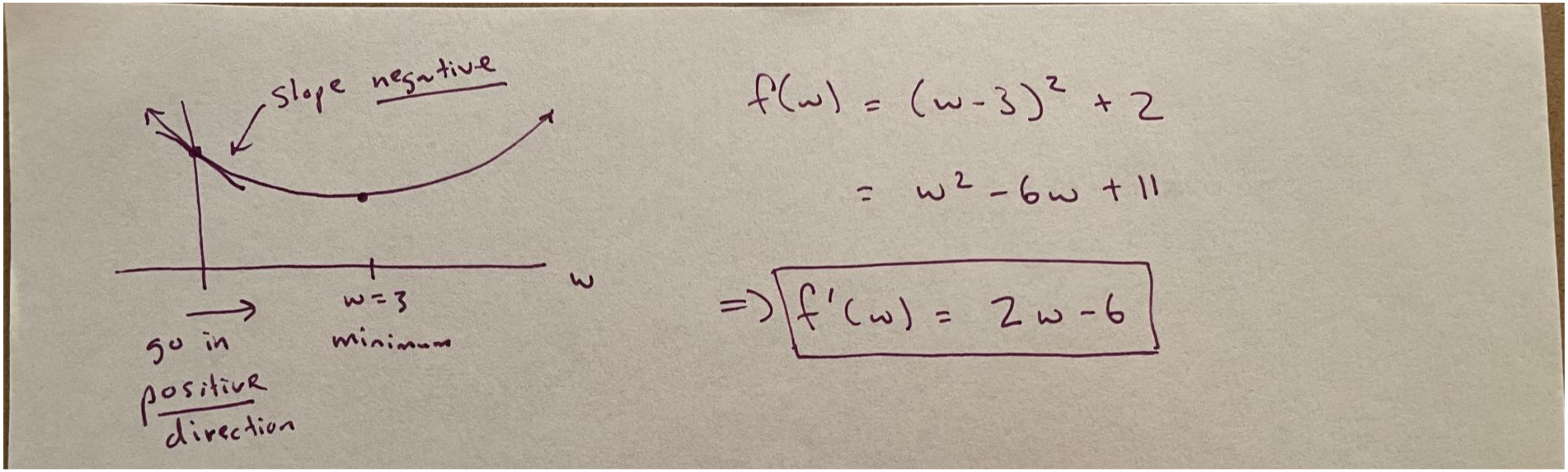
Stochastic gradient descent example

Goal: minimize the function $f(w) = w^2 - 6w + 11$



$$w \leftarrow w - \alpha f'(w)$$

step size

Stochastic gradient descent example

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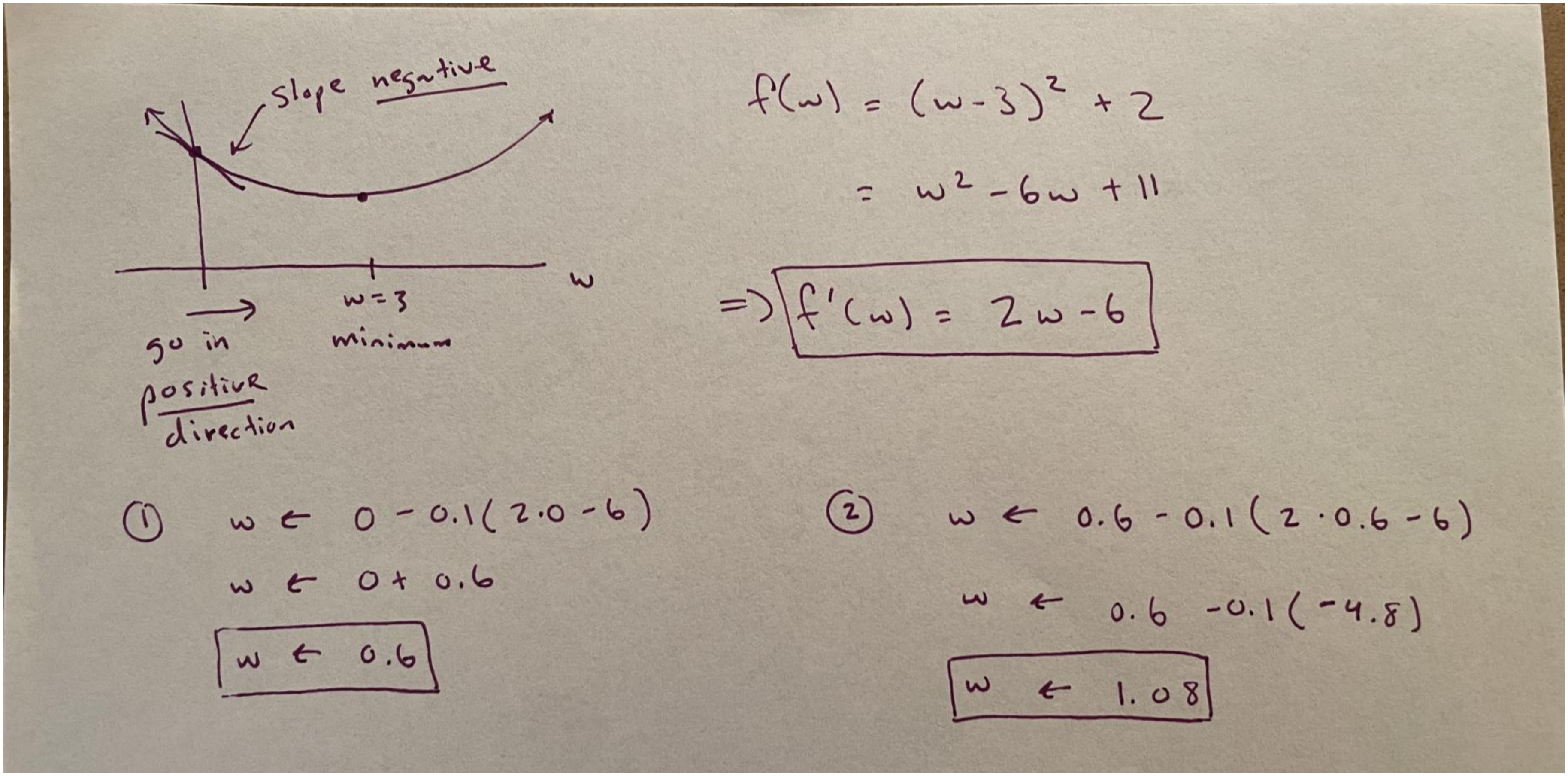

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Stochastic gradient descent example

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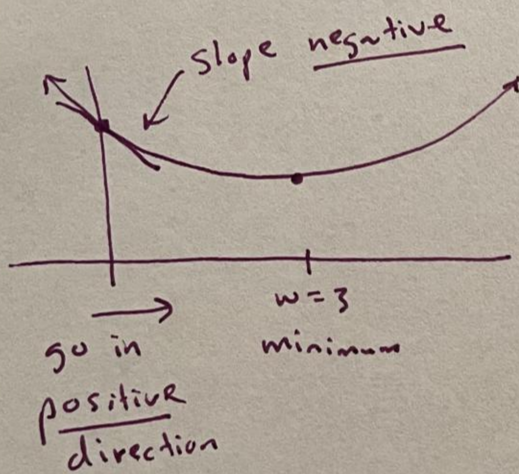


$$f(w) = (w-3)^2 + 2$$

$$= w^2 - 6w + 11$$

$$\Rightarrow \boxed{f'(w) = 2w - 6}$$

① $w \leftarrow 0 - 0.1(2 \cdot 0 - 6)$

$w \leftarrow 0 + 0.6$

$\boxed{w \leftarrow 0.6}$

② $w \leftarrow 0.6 - 0.1(2 \cdot 0.6 - 6)$

$w \leftarrow 0.6 - 0.1(-4.8)$

$\boxed{w \leftarrow 1.08}$

<u>Stop when:</u>

$$|f(w^t) - f(w^{t-1})| < \varepsilon, \quad \varepsilon = 1 \times 10^{-8}$$

(for example)

Stochastic Gradient Descent for Linear Regression

Key Idea: take the derivative of **one datapoint** at a time and use that to update w

$$J(\vec{w}) = \frac{1}{2} \sum_{i=1}^{n} (\vec{w} \cdot \vec{x}_i - y_i)^2$$

gradient with respect to <u>one</u> datapoint: (i.e. $\vec{x}_i$)

$$\nabla J_{\vec{x}_i} = \frac{\partial J(\vec{w})_{\vec{x}_i}}{\partial \vec{w}} = (\vec{w} \cdot \vec{x}_i - y_i)\, \vec{x}_i$$

Stochastic Gradient Descent for Linear Regression

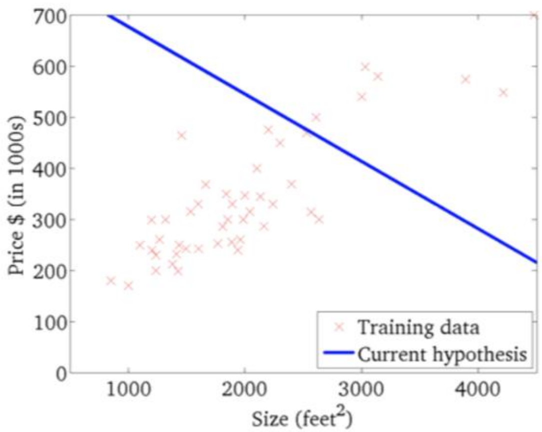
for (epoch) iteration t:

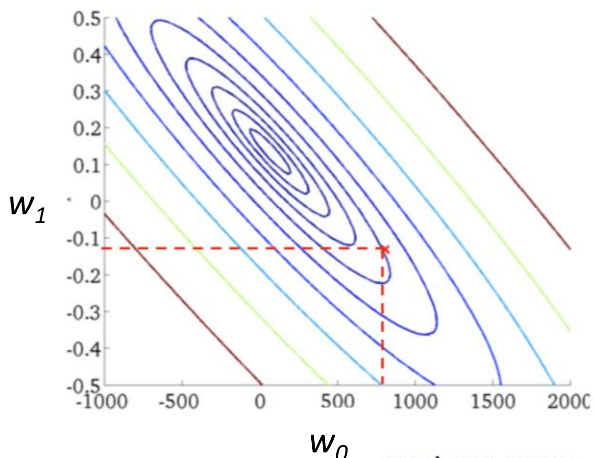
for $i = 1, 2, 3 \cdots n$: } usually shuffle

$$\vec{w} \leftarrow \vec{w} - \alpha(\vec{w} \cdot \vec{x}_i - y_i)\vec{x}_i$$

check for convergence: $|J(\vec{w}^t) - J(\vec{w}^{t-1})| < \varepsilon$

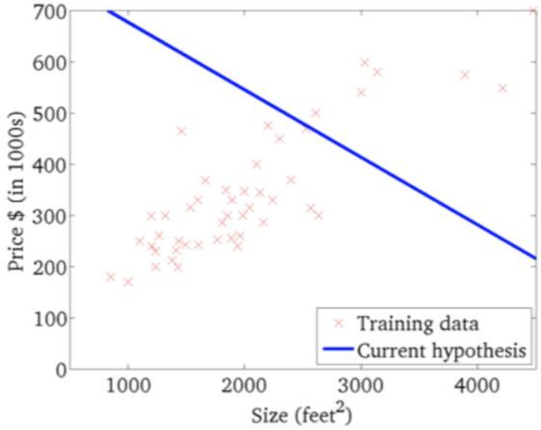
Linear Model and Cost Function J

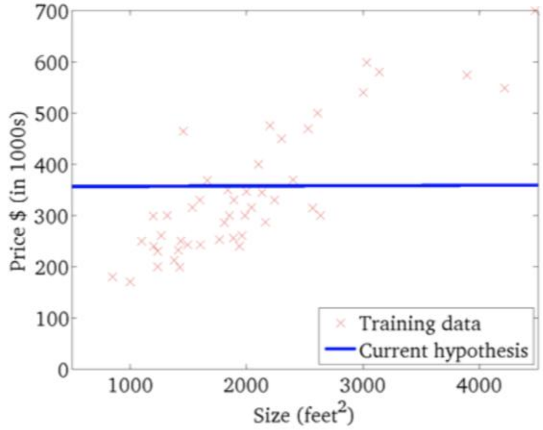


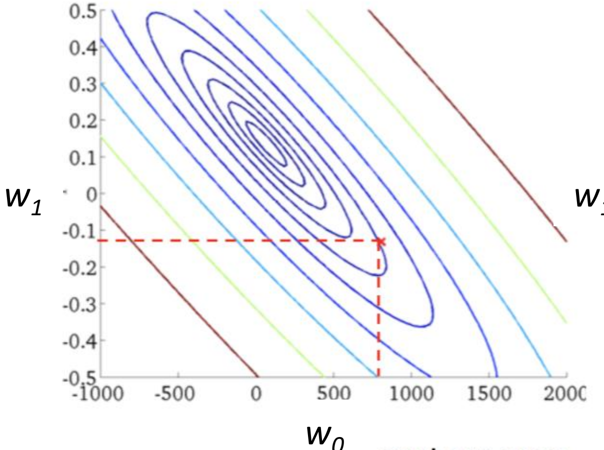


contour map

Linear Model and Cost Function J







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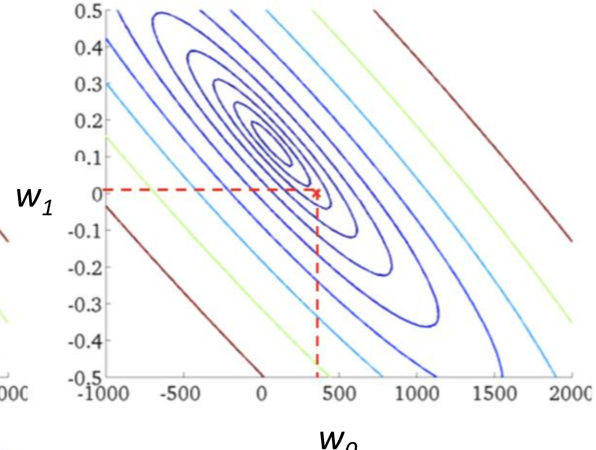
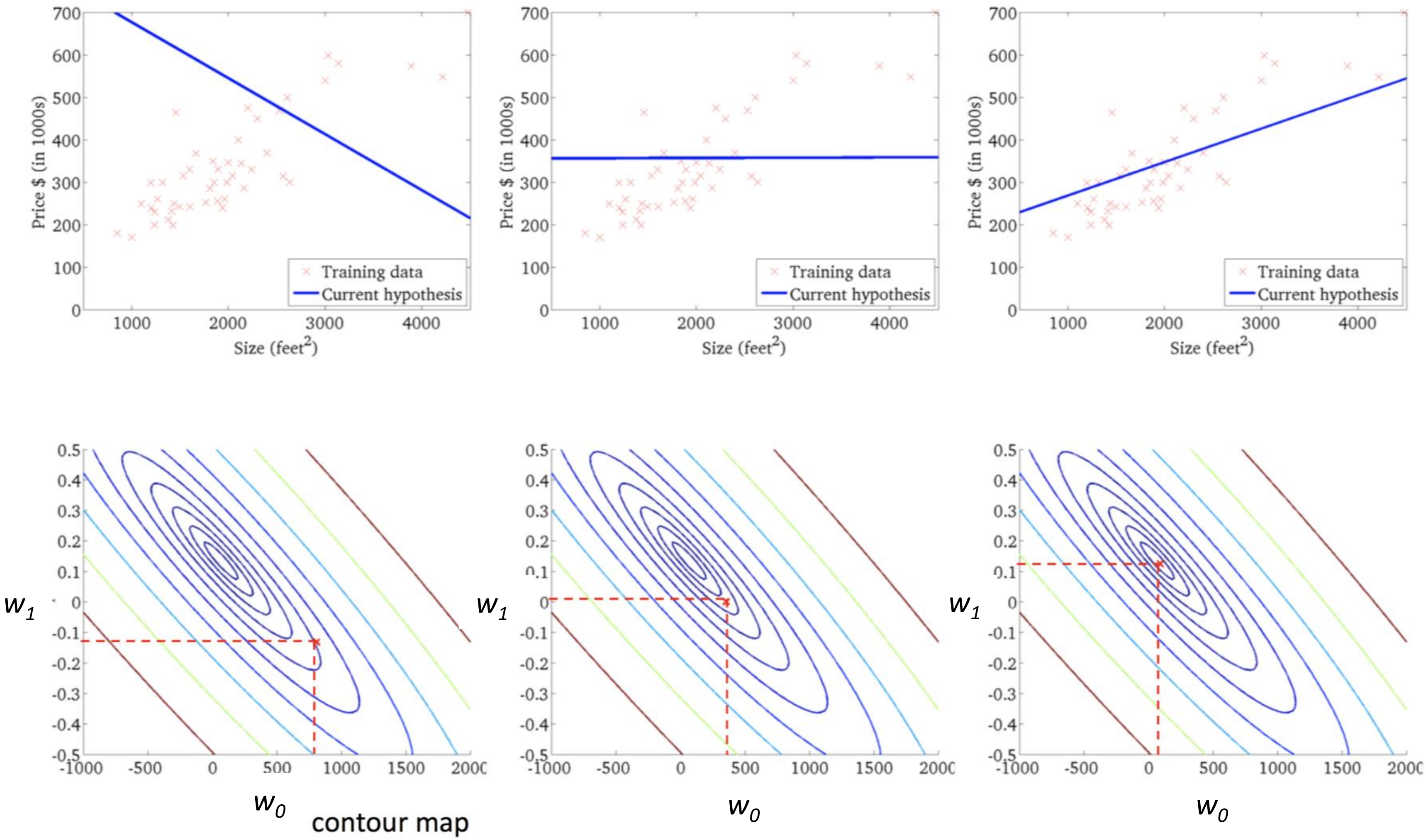


Slide: modified from Jessica Wu
Based on slide by Andrew Ng

Linear Model and Cost Function J



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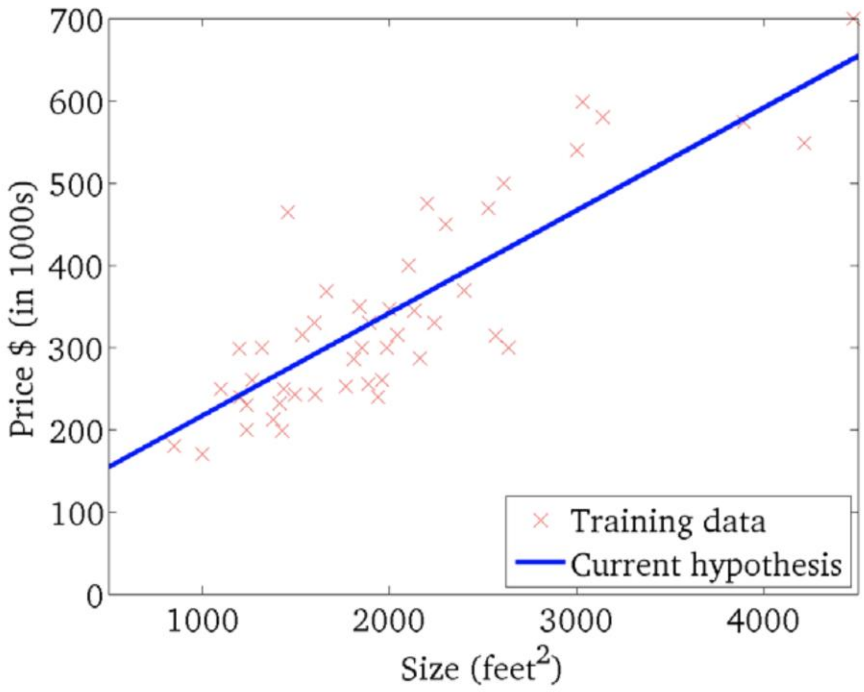
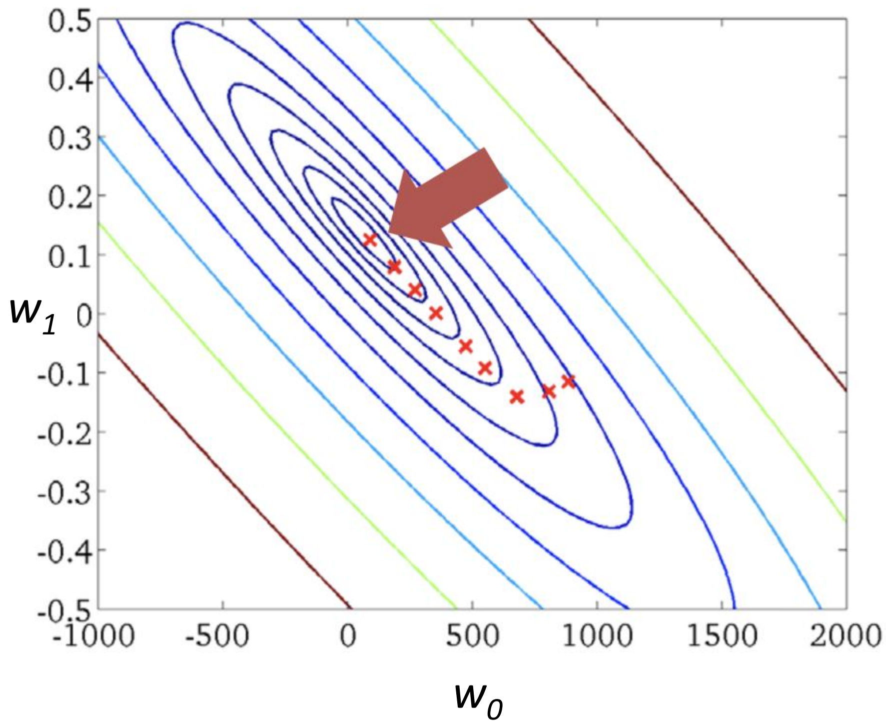
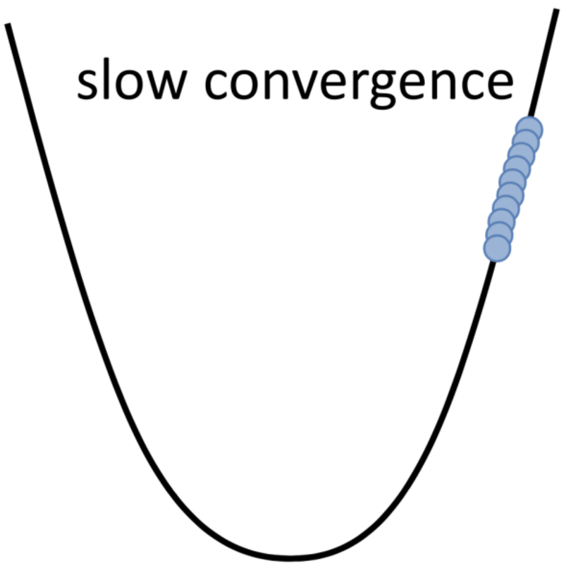
Gradient Descent: walking toward the minimum



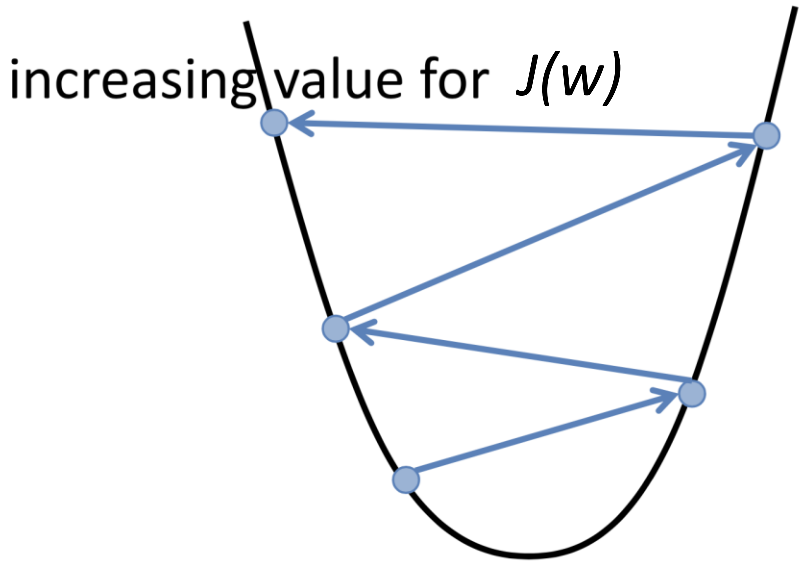


Choosing the step size alpha

α too small



α too large



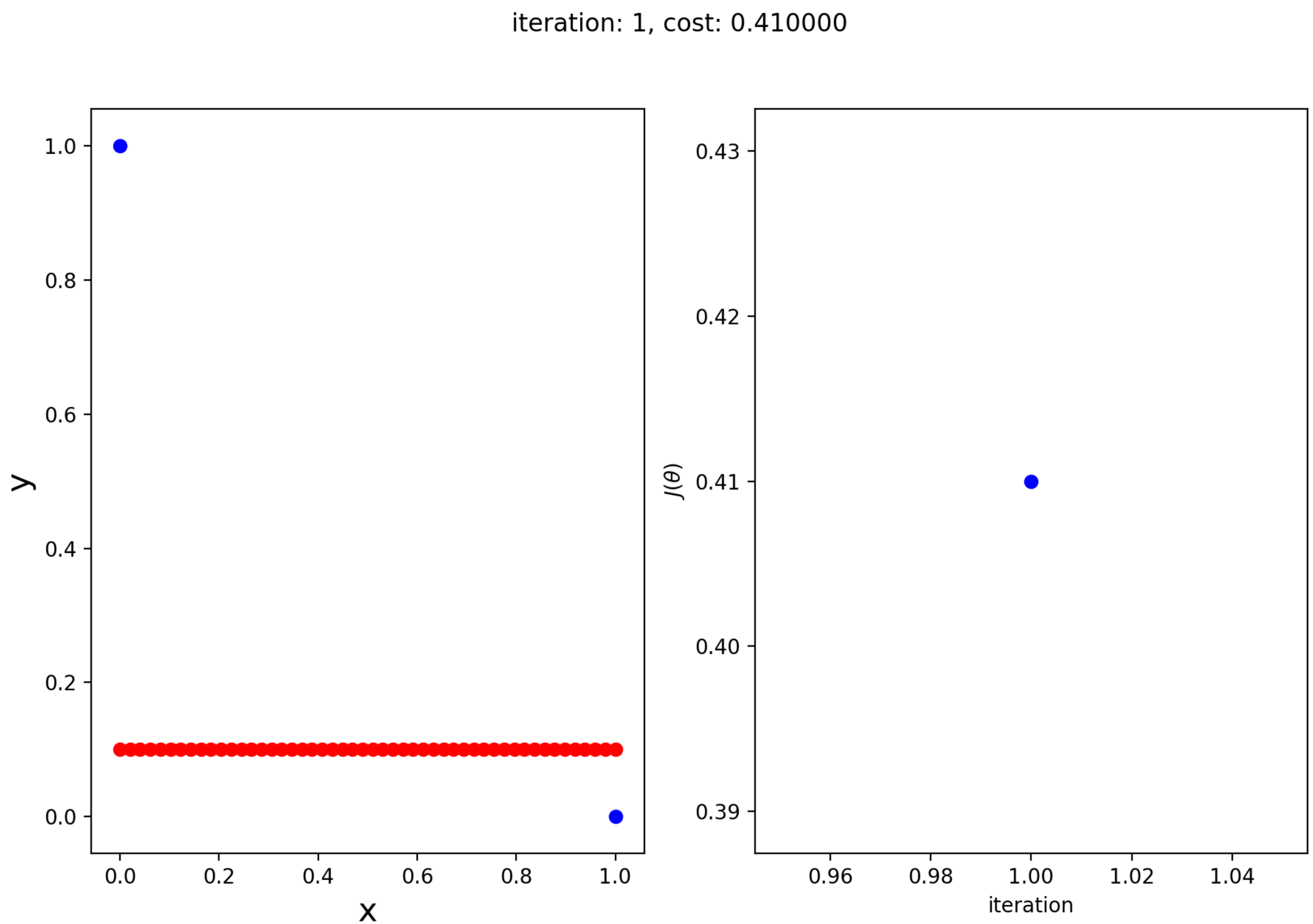
- may overshoot minimum
- may fail to converge (may even diverge)

Slide: modified from Jessica Wu
Based on slide by Eric Eaton
[Originally by Andrew Ng]

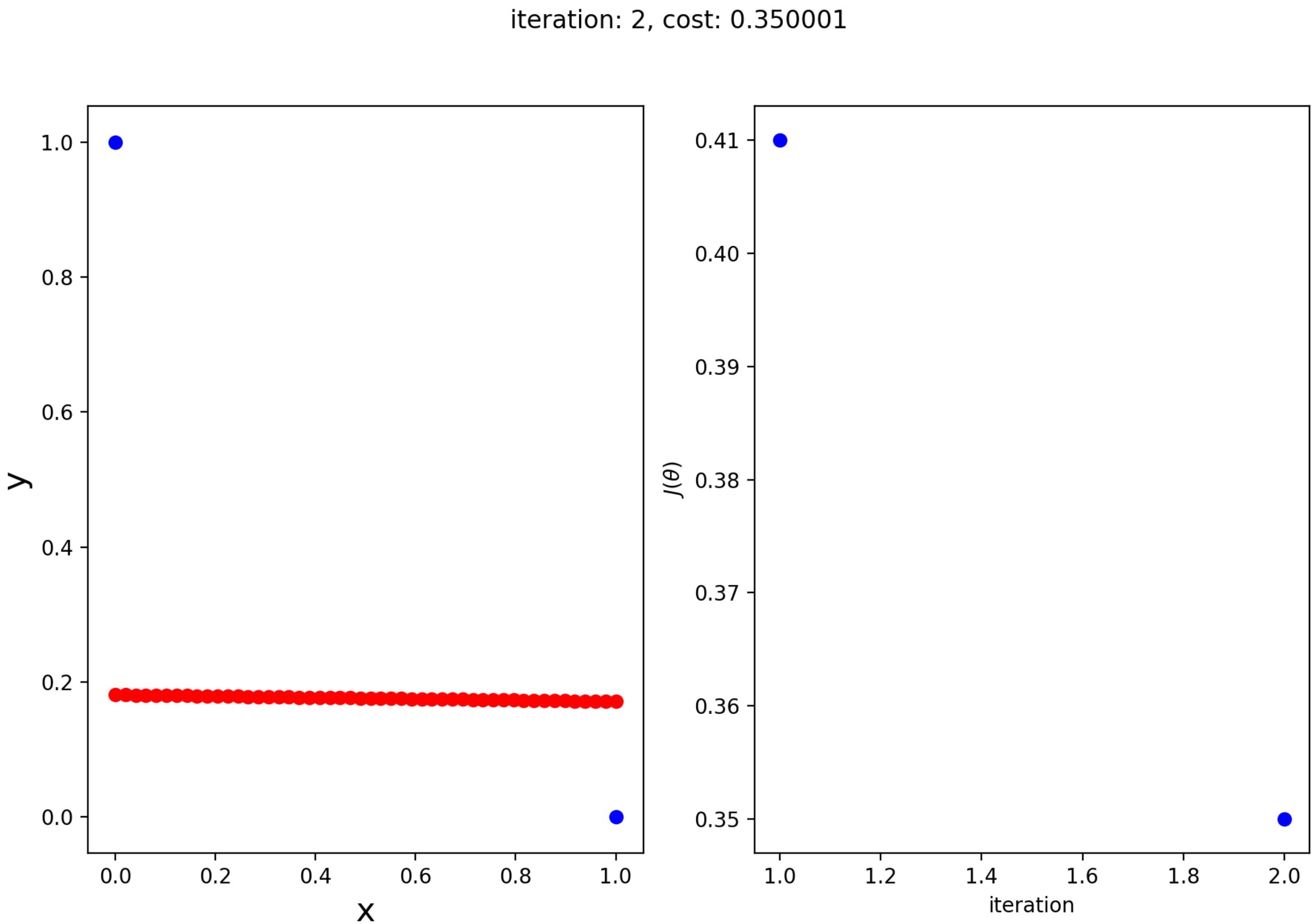
SGD with our small dataset from the handouts

Note: this is with the original order of the points

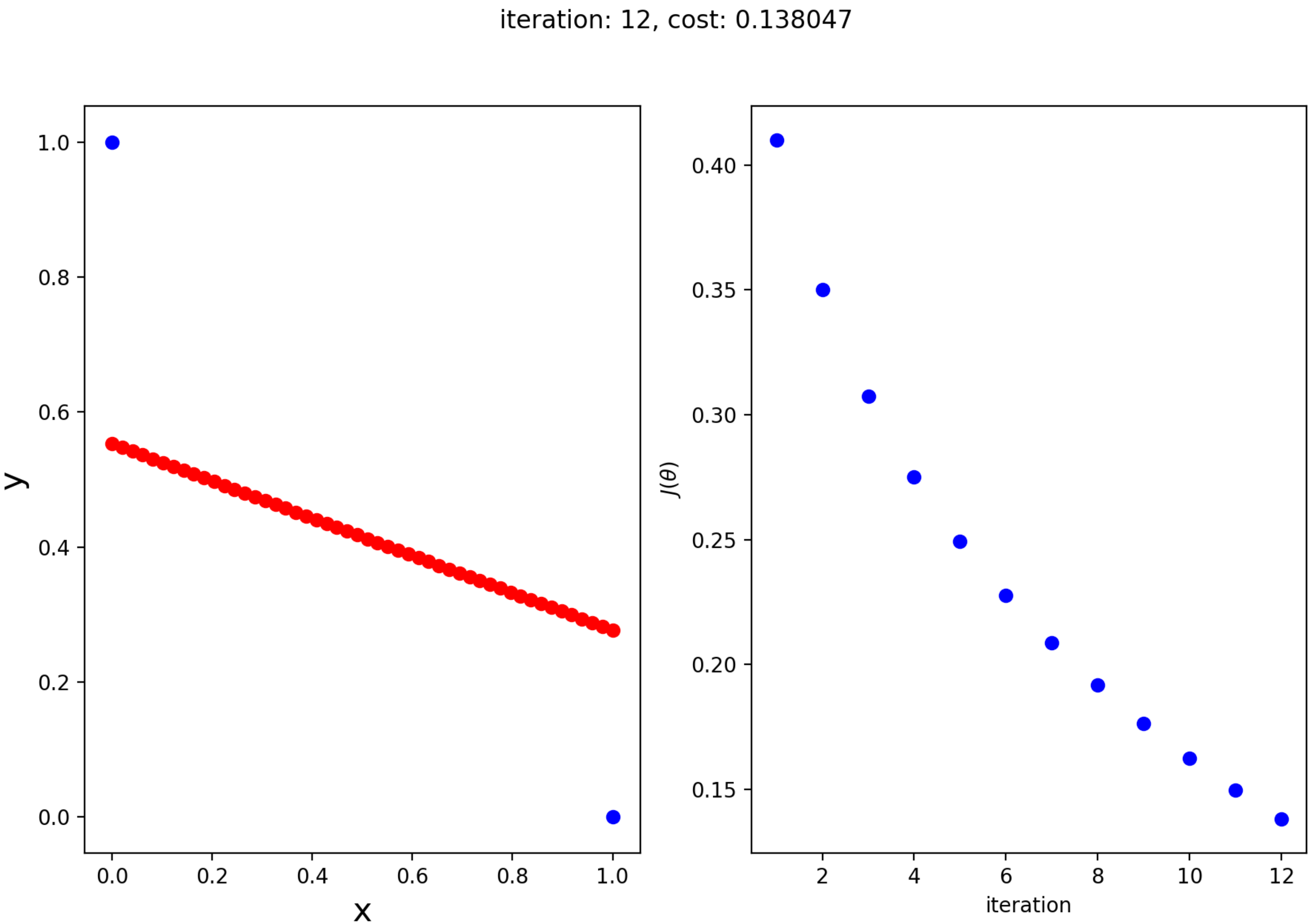
Small example, iteration 1



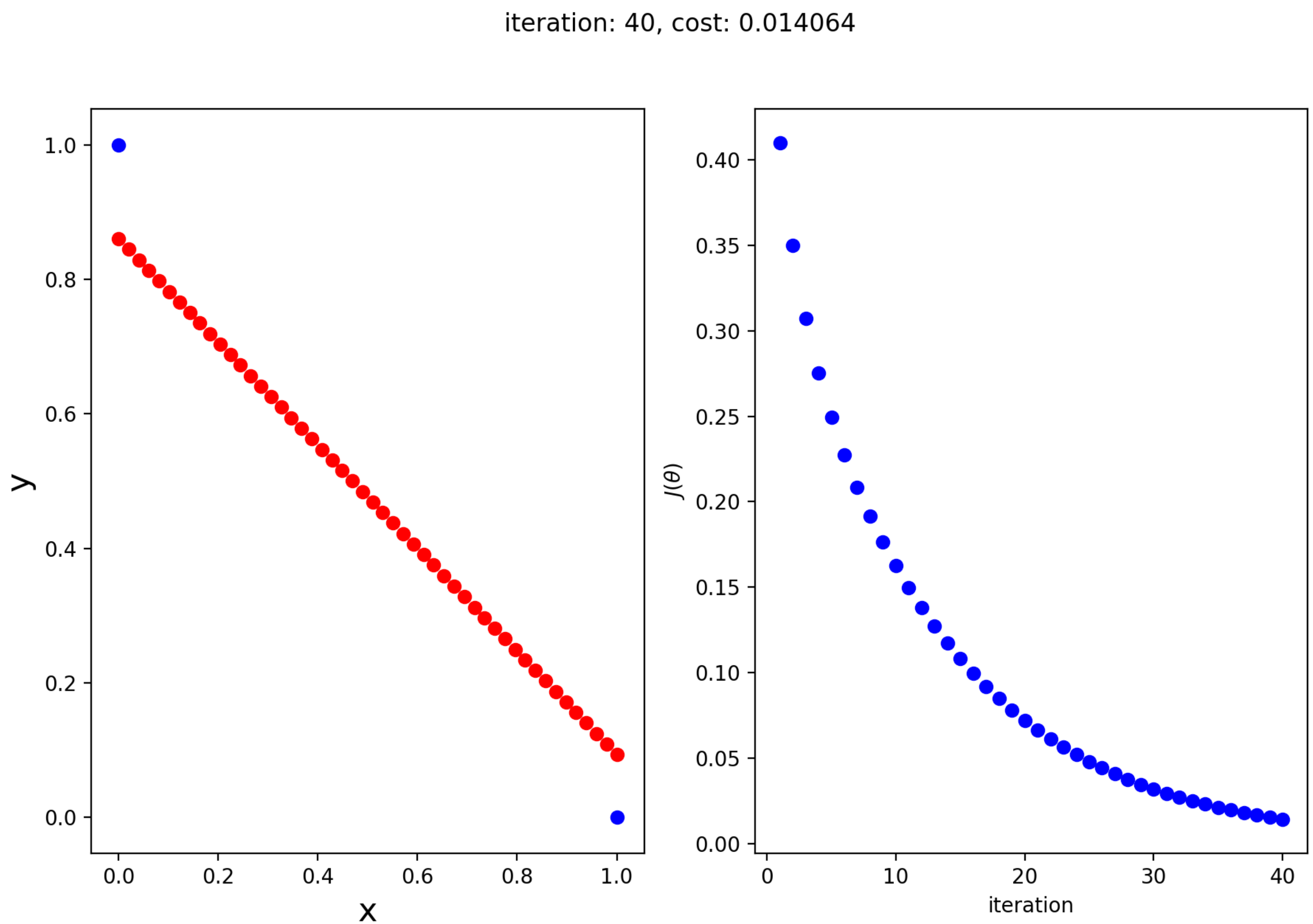
Small example, iteration 2



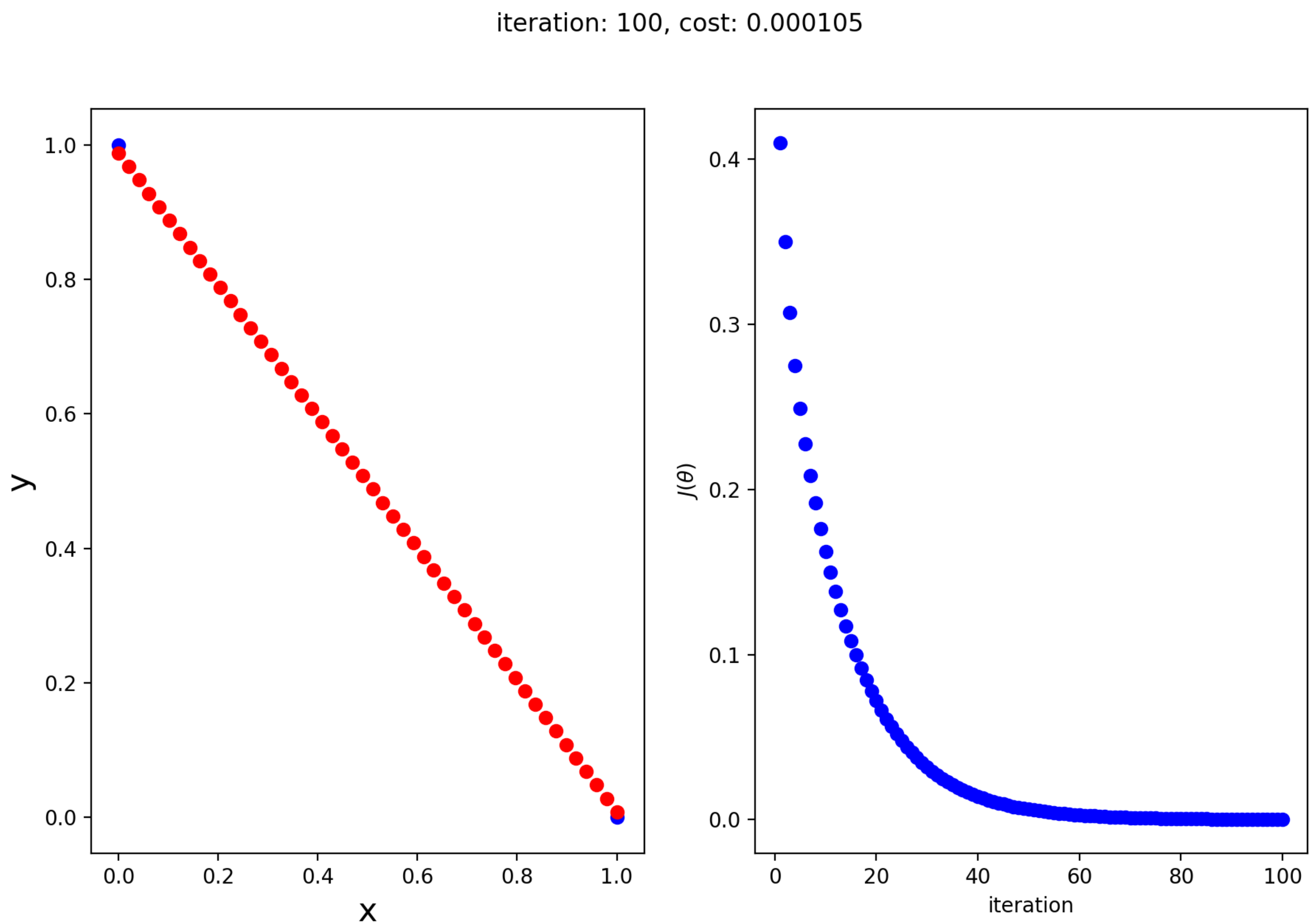
Small example, iteration 12



Small example, iteration 40



Small example, iteration 100



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Handout 6

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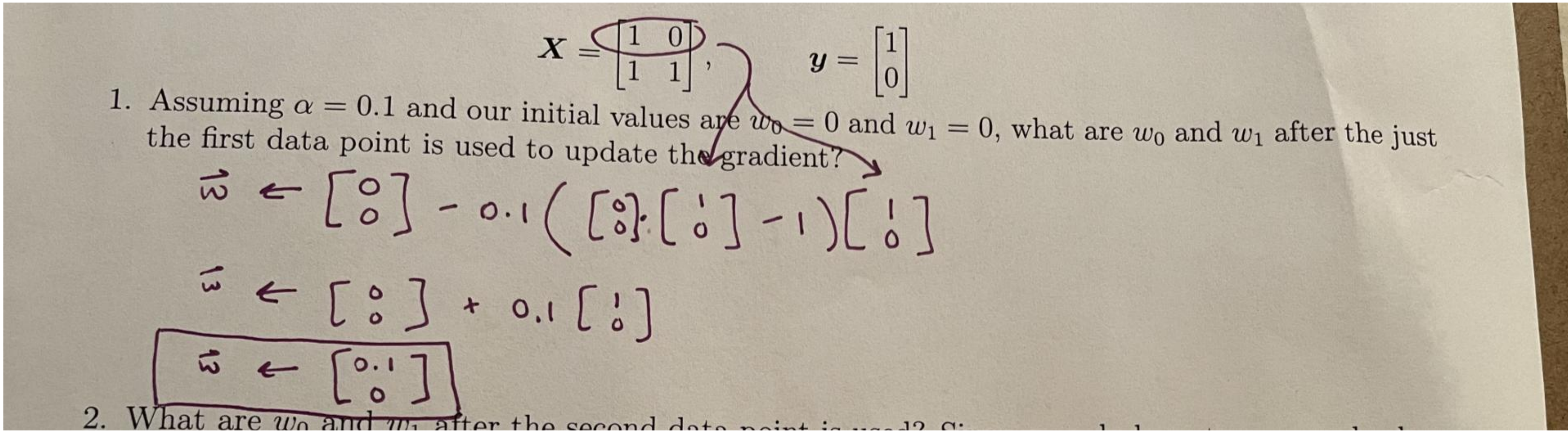

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

1. Assuming $\alpha = 0.1$ and our initial values are $w_0 = 0$ and $w_1 = 0$, what are w_0 and w_1 after the just the first data point is used to update the gradient?

$$\vec{w} \leftarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 0.1\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 1\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{w} \leftarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{w} \leftarrow \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

2. What are w_0 and w_1 after the second data point is used?

Handout 6

$$\boldsymbol{X} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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$$\vec{w} \leftarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{\vec{w} \leftarrow \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}}$$

2. What are w_0 and w_1 after the second data point is used? Since we only have two examples here, your result would be the weight vector after the first iteration of SGD.

$$\vec{w} \leftarrow \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} - 0.1\left(\begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 0\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{\vec{w} \leftarrow \begin{bmatrix} 0.09 \\ -0.01 \end{bmatrix}}$$

Handout 6

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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$$\boxed{\vec{w} \leftarrow \begin{bmatrix} 0.09 \\ -0.01 \end{bmatrix}}$$

3. What is the value of the objective function (cost) after this initial iteration?

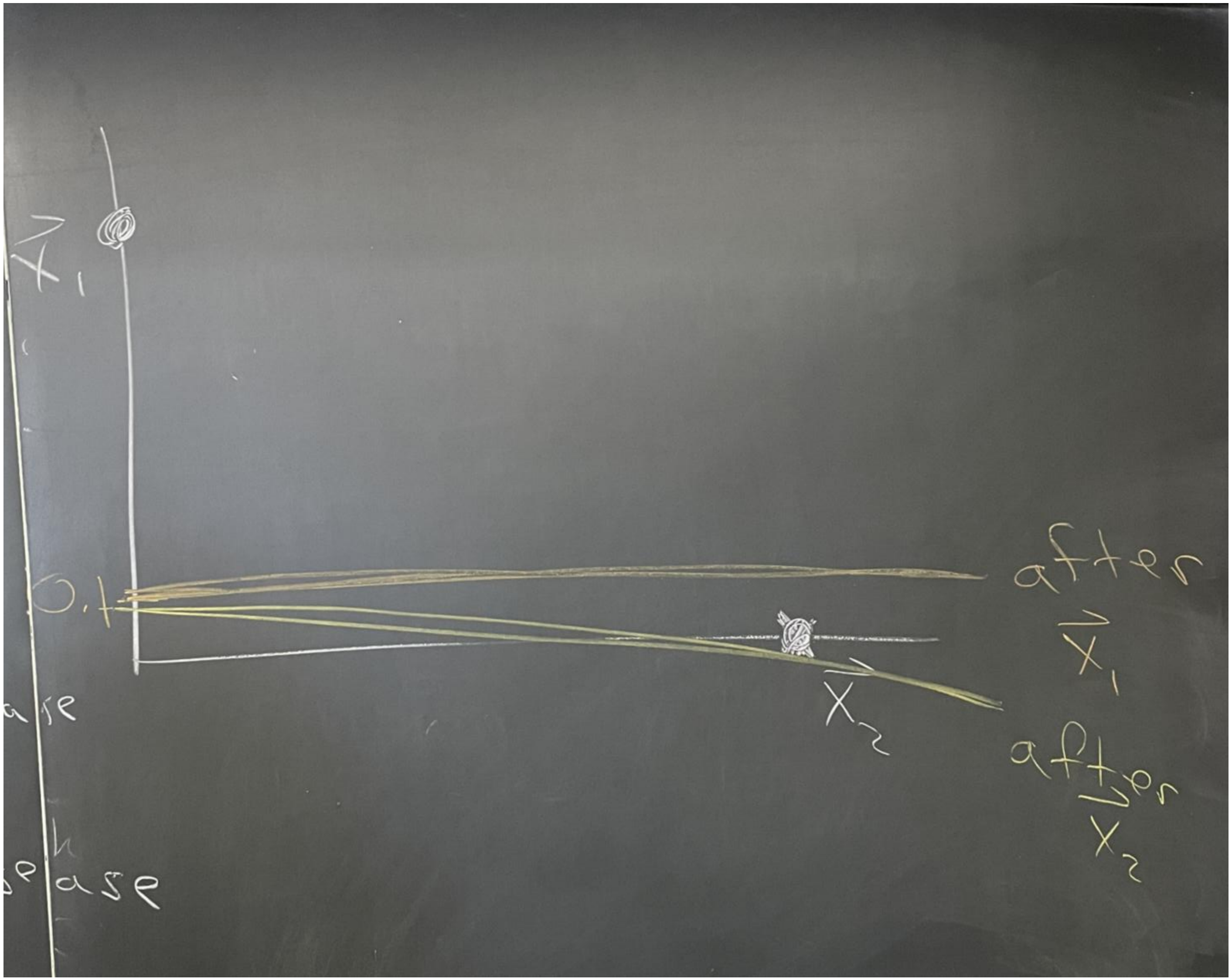
$$\hat{y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} 0.09 \\ -0.01 \end{bmatrix}}_{\vec{w}} = \underbrace{\begin{bmatrix} 0.09 \\ 0.08 \end{bmatrix}}_{\hat{y}} \quad \text{residuals}$$

$$\vec{y} - \hat{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0.09 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 0.91 \\ -0.08 \end{bmatrix}$$

$$J(\vec{w}) = \frac{1}{2} \begin{bmatrix} 0.91 \\ -0.08 \end{bmatrix} \cdot \begin{bmatrix} 0.91 \\ -0.08 \end{bmatrix}$$

$$\boxed{J(\vec{w}) = 0.417}$$

Handout 6 (#4)



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Pros and Cons

Gradient Descent

- requires multiple iterations
- need to choose α
- works well when p is large
- can support online learning

(Analytic Solution)

Normal Equations

- non-iterative
- no need for α
- slow if p is large
  - matrix inversion is $O(p^3)$

Slide: modified from Jessica Wu
Based on slide by Eric Eaton

Linear Regression Runtime

- T = # iterations of SGD
- n = # examples
- p = # features

1) What is the runtime of SGD?
2) What is the runtime of the analytic solution?

Challenge outside of class!

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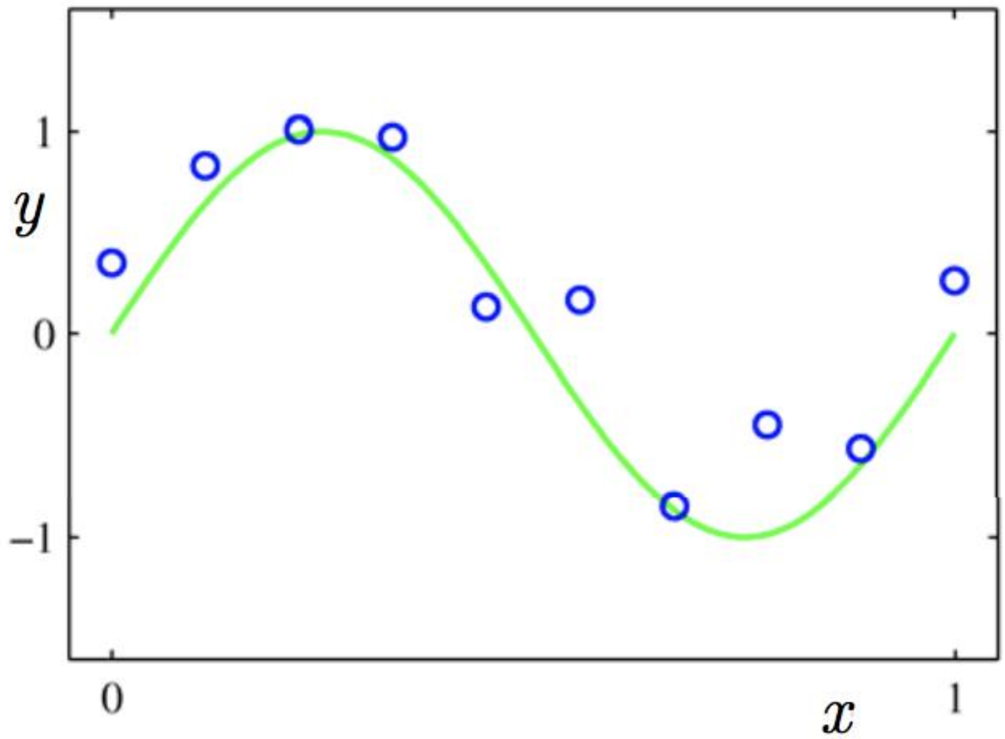
Polynomial Regression

- Can be thought of as regular linear regression with a change of basis



Plot: Jessica Wu

Polynomial Regression

$$\boldsymbol{X} = \begin{bmatrix} x_1^0 & x_1^1 & x_1^2 & \cdots & x_1^d \\ & & \vdots & & \\ x_n^0 & x_n^1 & x_n^2 & \cdots & x_n^d \end{bmatrix}$$